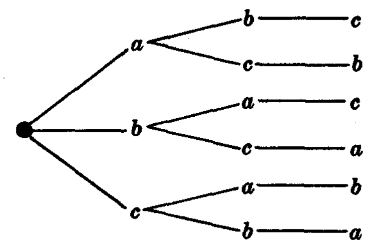
[09-04-26-SO11]

TREE DIAGRAMS

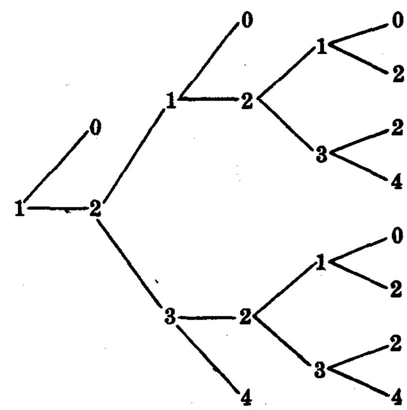
2.29. Construct the tree diagram for the number of permutations of $\{a, b, c\}$.



The six permutations are listed on the right of the diagram.

2.30. A man has time to play roulette at most five times. At each play he wins or loses a dollar. The man begins with one dollar and will stop playing before the five times if he loses all his money or if he wins three dollars, i.e. if he has four dollars. Find the number of ways that the betting can occur.

The tree diagram on the right describes the way the betting can occur. Each number in the diagram denotes the number of dollars the man has at that point. Observe that the betting can occur in 11 different ways. Note that he will stop betting before the five times are up in only three of the cases.



Supplementary Problems

FACTORIAL

2.31. Compute: (i) $9!$, (ii) $10!$, (iii) $11!$

2.32. Compute: (i) $\frac{16!}{14!}$, (ii) $\frac{14!}{11!}$, (iii) $\frac{8!}{10!}$, (iv) $\frac{10!}{13!}$.

2.33. Simplify: (i) $\frac{(n+1)!}{n!}$, (ii) $\frac{n!}{(n-2)!}$, (iii) $\frac{(n-1)!}{(n+2)!}$, (iv) $\frac{(n-r+1)!}{(n-r-1)!}$.

PERMUTATIONS

2.34. (i) How many automobile license plates can be made if each plate contains 2 different letters followed by 3 different digits? (ii) Solve the problem if the first digit cannot be 0.

2.35. There are 6 roads between A and B and 4 roads between B and C.

(i) In how many ways can one drive from A to C by way of B?

(ii) In how many ways can one drive roundtrip from A to C by way of B?

(iii) In how many ways can one drive roundtrip from A to C without using the same road more than once?

2.36. Find the number of ways in which 6 people can ride a toboggan if one of three must drive.

2.37. (i) Find the number of ways in which five persons can sit in a row.
(ii) How many ways are there if two of the persons insist on sitting next to one another?

2.38. Solve the preceding problem if they sit around a circular table.

2.39. (i) Find the number of four letter words that can be formed from the letters of the word HISTORY. (ii) How many of them contain only consonants? (iii) How many of them begin and end in a consonant? (iv) How many of them begin with a vowel? (v) How many contain the letter Y? (vi) How many begin with T and end in a vowel? (vii) How many begin with T and also contain S? (viii) How many contain both vowels?

2.40. How many different signals, each consisting of 8 flags hung in a vertical line, can be formed from 4 red flags, 2 blue flags and 2 green flags?

2.41. Find the number of permutations that can be formed from all the letters of each word: (i) queue, (ii) committee, (iii) proposition, (iv) baseball.

2.42. (i) Find the number of ways in which 4 boys and 4 girls can be seated in a row if the boys and girls are to have alternate seats.
(ii) Find the number of ways if they sit alternately and if one boy and one girl are to sit in adjacent seats.
(iii) Find the number of ways if they sit alternately and if one boy and one girl must not sit in adjacent seats.

2.43. Solve the preceding problem if they sit around a circular table.

2.44. An urn contains 10 balls. Find the number of ordered samples (i) of size 3 with replacement, (ii) of size 3 without replacement, (iii) of size 4 with replacement, (iv) of size 5 without replacement.

2.45. Find the number of ways in which 5 large books, 4 medium-size books and 3 small books can be placed on a shelf so that all books of the same size are together.

2.46. Consider all positive integers with 3 different digits. (Note that 0 cannot be the first digit.) (i) How many are greater than 700? (ii) How many are odd? (iii) How many are even? (iv) How many are divisible by 5?

2.47. (i) Find the number of distinct permutations that can be formed from all of the letters of the word ELEVEN. (ii) How many of them begin and end with E? (iii) How many of them have the 3 E's together? (iv) How many begin with E and end with N?

BINOMIAL COEFFICIENTS AND THEOREM

2.48. Compute: (i) $\binom{5}{2}$, (ii) $\binom{7}{3}$, (iii) $\binom{14}{2}$, (iv) $\binom{6}{4}$, (v) $\binom{20}{17}$, (vi) $\binom{18}{15}$.

2.49. Compute: (i) $\binom{9}{3,\,5,\,1}$, (ii) $\binom{7}{3,\,2,\,2,\,0}$, (iii) $\binom{6}{2,\,2,\,1,\,1,\,0}$.

2.50. Expand and simplify: (i) $(2x+y^2)^3$, (ii) $(x^2-3y)^4$, (iii) $(\frac{1}{2}a+2b)^5$, (iv) $(2a^2-b)^6$.

2.51. Show that $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n} = 2^n$.

2.52. Show that $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots\pm\binom{n}{n} = 0$.

2.53. Find the term in the expansion of $(2x^2-\frac{1}{2}y^3)^8$ which contains x^8.

2.54. Find the term in the expansion of $(3xy^2-z^2)^7$ which contains y^6.

COMBINATIONS

2.55. A class contains 9 boys and 3 girls. (i) In how many ways can the teacher choose a committee of 4? (ii) How many of them will contain at least one girl? (iii) How many of them will contain exactly one girl?

2.56. A woman has 11 close friends. (i) In how many ways can she invite 5 of them to dinner? (ii) In how many ways if two of the friends are married and will not attend separately? (iii) In how many ways if two of them are not on speaking terms and will not attend together?

2.57. There are 10 points $A, B, \ldots$ in a plane, no three on the same line. (i) How many lines are determined by the points? (ii) How many of these lines do not pass through A or B? (iii) How many triangles are determined by the points? (iv) How many of these triangles contain the point A? (v) How many of these triangles contain the side AB?

2.58. A student is to answer 10 out of 13 questions on an exam. (i) How many choices has he? (ii) How many if he must answer the first two questions? (iii) How many if he must answer the first or second question but not both? (iv) How many if he must answer exactly 3 of the first 5 questions? (v) How many if he must answer at least 3 of the first 5 questions?

2.59. A man is dealt a poker hand (5 cards) from an ordinary playing deck. In how many ways can he be dealt (i) a straight flush, (ii) four of a kind, (iii) a straight, (iv) a pair of aces, (v) two of a kind (a pair)?

2.60. The English alphabet has 26 letters of which 5 are vowels.

(i) How many 5 letter words containing 3 different consonants and 2 different vowels can be formed?
(ii) How many of them contain the letter b?
(iii) How many of them contain the letters b and c?
(iv) How many of them begin with b and contain the letter c?
(v) How many of them begin with b and end with c?
(vi) How many of them contain the letters a and b?
(vii) How many of them begin with a and contain b?
(viii) How many of them begin with b and contain a?
(ix) How many of them begin with a and end with b?
(x) How many of them contain the letters a, b and c?

ORDERED AND UNORDERED PARTITIONS

2.61. In how many ways can 9 toys be divided evenly among 3 children?

2.62. In how many ways can 9 students be evenly divided into three teams?

2.63. In how many ways can 10 students be divided into three teams, one containing 4 students and the others 3?

2.64. There are 12 balls in an urn. In how many ways can 3 balls be drawn from the urn, four times in succession, all without replacement?

2.65. In how many ways can a club with 12 members be partitioned into three committees containing 5, 4 and 3 members respectively?

2.66. In how many ways can n students be partitioned into two teams containing at least one student?

2.67. In how many ways can 14 men be partitioned into 6 committees where 2 of the committees contain 3 men and the others 2?

TREE DIAGRAMS

2.68. Construct the tree diagram for the number of permutations of $\{a, b, c, d\}$.

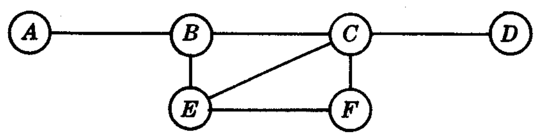
2.69. Find the product set $\{1, 2, 3\} \times \{2, 4\} \times \{2, 3, 4\}$ by constructing the appropriate tree diagram.

2.70. Teams A and B play in a basketball tournament. The first team that wins two games in a row or a total of four games wins the tournament. Find the number of ways the tournament can occur.

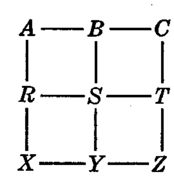
2.71. A man has time to play roulette five times. He wins or loses a dollar at each play. The man begins with two dollars and will stop playing before the five times if he loses all his money or wins three dollars (i.e. has five dollars). Find the number of ways the playing can occur.

2.72. A man is at the origin on the x-axis and takes a unit step either to the left or to the right. He stops after 5 steps or if he reaches 3 or -2. Construct the tree diagram to describe all possible paths the man can travel.

2.73. In the following diagram let $A, B, \ldots, F$ denote islands, and the lines connecting them bridges. A man begins at A and walks from island to island. He stops for lunch when he cannot continue to walk without crossing the same bridge twice. Find the number of ways that he can take his walk before eating lunch.



2.74. Consider the adjacent diagram with nine points $A, B, C, R, S, T, X, Y, Z$. A man begins at X and is allowed to move horizontally or vertically, one step at a time. He stops when he cannot continue to walk without reaching the same point more than once. Find the number of ways he can take his walk, if he first moves from X to R. (By symmetry, the total number of ways is twice this.)



Answers to Supplementary Problems

2.31. (i) 362,880 (ii) 3,628,800 (iii) 39,916,800

2.32. (i) 240 (ii) 2184 (iii) 1/90 (iv) 1/1716

2.33. (i) $n+1$ (ii) $n(n-1) = n^2 - n$ (iii) $1/[n(n+1)(n+2)]$ (iv) $(n-r)(n-r+1)$

2.34. (i) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468{,}000$ (ii) $26 \cdot 25 \cdot 9 \cdot 9 \cdot 8 = 421{,}200$

2.35. (i) $6 \cdot 4 = 24$ (ii) $6 \cdot 4 \cdot 4 \cdot 6 = 24 \cdot 24 = 576$ (iii) $6 \cdot 4 \cdot 3 \cdot 5 = 360$

2.36. $3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 360$

2.37. (i) $5! = 120$ (ii) $4 \cdot 2! \cdot 3! = 48$

2.38. (i) $4! = 24$ (ii) $2!\,3! = 12$

2.39. (i) $7 \cdot 6 \cdot 5 \cdot 4 = 840$ (ii) $5 \cdot 4 \cdot 3 \cdot 2 = 120$ (iii) $5 \cdot 5 \cdot 4 \cdot 4 = 400$ (iv) $2 \cdot 6 \cdot 5 \cdot 4 = 240$ (v) $4 \cdot 6 \cdot 5 \cdot 4 = 480$ (vi) $1 \cdot 5 \cdot 4 \cdot 2 = 40$ (vii) $1 \cdot 3 \cdot 5 \cdot 4 = 60$ (viii) $4 \cdot 3 \cdot 5 \cdot 4 = 240$

2.40. $\dfrac{8!}{4!\,2!\,2!} = 420$

2.41. (i) $\dfrac{5!}{2!\,2!} = 30$ (ii) $\dfrac{9!}{2!\,2!\,2!} = 45{,}360$ (iii) $\dfrac{11!}{2!\,3!\,2!} = 1{,}663{,}200$ (iv) $\dfrac{8!}{2!\,2!\,2!} = 5040$

2.42. (i) $2 \cdot 4! \cdot 4! = 1152$ (ii) $2 \cdot 7 \cdot 3! \cdot 3! = 504$ (iii) $1152 - 504 = 648$

2.43. (i) $3! \cdot 4! = 144$ (ii) $2 \cdot 3! \cdot 3! = 72$ (iii) $144 - 72 = 72$

2.44. (i) $10 \cdot 10 \cdot 10 = 1000$ (iii) $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$
(ii) $10 \cdot 9 \cdot 8 = 720$ (iv) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$

2.45. $3!\,5!\,4!\,3! = 103{,}680$

2.46. (i) $3 \cdot 9 \cdot 8 = 216$ (ii) $8 \cdot 8 \cdot 5 = 320$
(iii) $9 \cdot 8 \cdot 1 = 72$ end in 0, and $8 \cdot 8 \cdot 4 = 256$ end in the other even digits; hence, altogether, $72 + 256 = 328$ are even.
(iv) $9 \cdot 8 \cdot 1 = 72$ end in 0, and $8 \cdot 8 \cdot 1 = 64$ end in 5; hence, altogether, $72 + 64 = 136$ are divisible by 5.

2.47. (i) $\frac{6!}{3!} = 120$ (ii) $4! = 24$ (iii) $4 \cdot 3! = 24$ (iv) $\frac{4!}{2!} = 12$

2.48. (i) 10 (ii) 35 (iii) 91 (iv) 15 (v) 1140 (vi) 816

2.49. (i) 504 (ii) 210 (iii) 180

2.50. (i) $8x^3 + 12x^2y^2 + 6xy^4 + y^6$
(ii) $x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4$
(iii) $a^5/32 + 5a^4b/8 + 5a^3b^2 + 20a^2b^3 + 40ab^4 + 32b^5$
(iv) $64a^{12} - 192a^{10}b + 240a^8b^2 - 160a^6b^3 + 60a^4b^4 - 12a^2b^5 + b^6$

2.51. *Hint.* Expand $(1+1)^n$.

2.52. *Hint.* Expand $(1-1)^n$.

2.53. $70x^8y^{12}$

2.54. $945x^3y^6z^8$

2.55. (i) $\binom{12}{4} = 495$, (ii) $\binom{12}{4} - \binom{9}{4} = 369$, (iii) $3 \cdot \binom{9}{3} = 252$

2.56. (i) $\binom{11}{5} = 462$, (ii) $\binom{9}{3} + \binom{9}{5} = 210$, (iii) $\binom{9}{5} + 2 \cdot \binom{9}{4} = 378$

2.57. (i) $\binom{10}{2} = 45$, (ii) $\binom{8}{2} = 28$, (iii) $\binom{10}{3} = 120$, (iv) $\binom{9}{2} = 36$, (v) 8

2.58. (i) $\binom{13}{10} = \binom{13}{3} = 286$ (iv) $\binom{5}{3}\binom{8}{7} = 80$
(ii) $\binom{11}{8} = \binom{11}{3} = 165$ (v) $\binom{5}{3}\binom{8}{7} + \binom{5}{4}\binom{8}{6} + \binom{5}{5}\binom{8}{5} = 276$
(iii) $2 \cdot \binom{11}{9} = 2 \cdot \binom{11}{2} = 110$

2.59. (i) $4 \cdot 10 = 40$, (ii) $13 \cdot 48 = 624$, (iii) $10 \cdot 4^5 - 40 = 10{,}200$. (We subtract the number of straight flushes.) (iv) $\binom{4}{2}\binom{12}{3} \cdot 4^3 = 84{,}480$, (v) $13 \cdot \binom{4}{2}\binom{12}{3} \cdot 4^3 = 1{,}098{,}240$

2.60. (i) $\binom{21}{3}\binom{5}{2} \cdot 5! = 1{,}596{,}000$ (v) $19 \cdot \binom{5}{2} \cdot 3! = 1140$ (ix) $4 \cdot \binom{20}{2} \cdot 3! = 4560$
(ii) $\binom{20}{2}\binom{5}{2} \cdot 5! = 228{,}000$ (vi) $4 \cdot \binom{20}{2} \cdot 5! = 91{,}200$ (x) $4 \cdot 19 \cdot 5! = 9120$
(iii) $19 \cdot \binom{5}{2} \cdot 5! = 22{,}800$ (vii) $4 \cdot \binom{20}{2} \cdot 4! = 18{,}240$
(iv) $19 \cdot \binom{5}{2} \cdot 4! = 4560$ (viii) 18,240 (same as (vii))

2.61. $\dfrac{9!}{3!\,3!\,3!} = 1680$

2.62. $1680/3! = 280$ or $\dbinom{8}{2}\dbinom{5}{2} = 280$

2.63. $\dfrac{10!}{4!\,3!\,3!}\cdot\dfrac{1}{2!} = 2100$ or $\dbinom{10}{4}\dbinom{5}{2} = 2100$

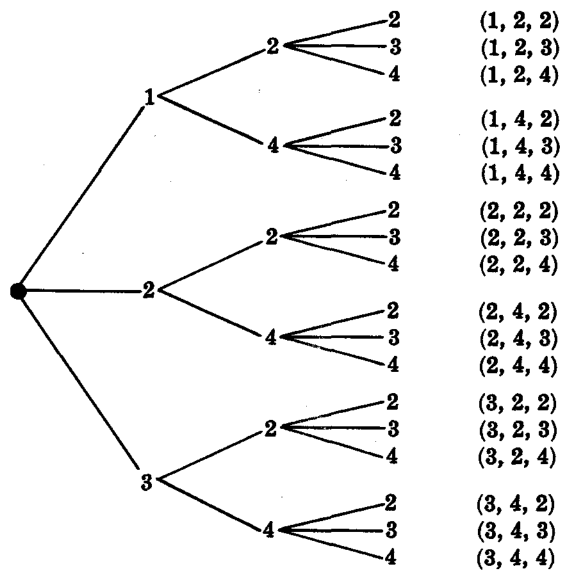
2.64. $\dfrac{12!}{3!\,3!\,3!\,3!} = 369{,}600$

2.65. $\dfrac{12!}{5!\,4!\,3!} = 27{,}720$

2.66. $2^{n-1} - 1$

2.67. $\dfrac{14!}{3!\,3!\,2!\,2!\,2!\,2!}\cdot\dfrac{1}{2!\,4!} = 3{,}153{,}150$

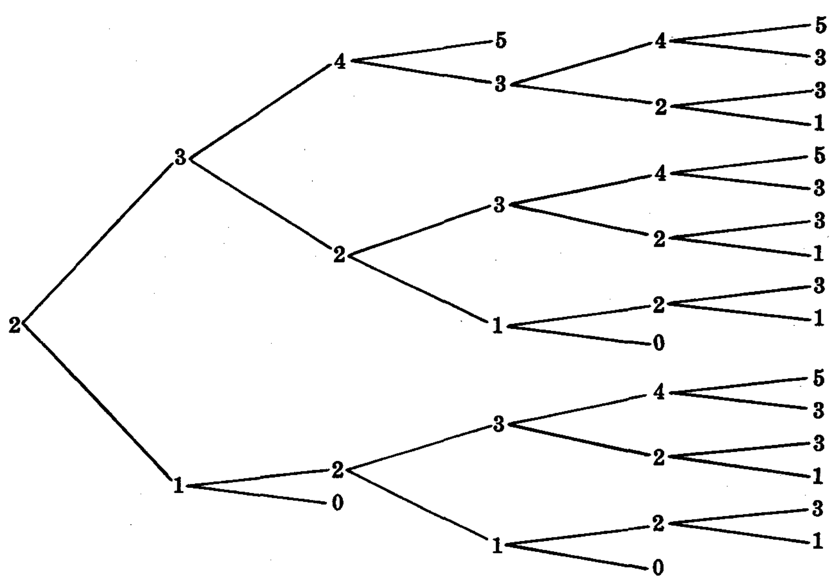
2.69.



The eighteen elements of the product set are listed to the right of the tree diagram.

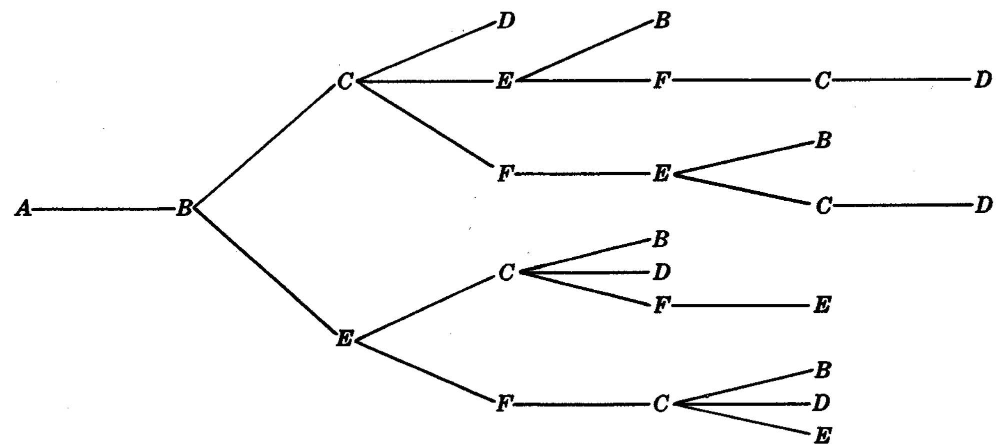
2.70. 14 ways

2.71. 20 ways (as seen in the following diagram):



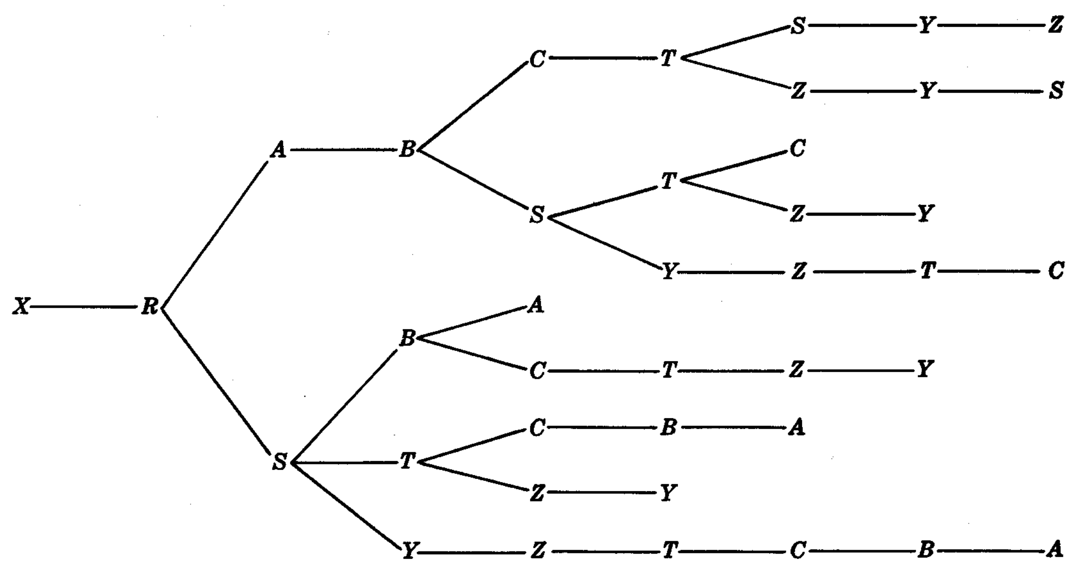
2.72. *Hint.* The tree is essentially the same as the tree of the preceding problem.

2.73. The appropriate tree diagram follows:



There are eleven ways to take his walk. Observe that he must eat his lunch at either *B*, *D* or *E*.

2.74. The appropriate tree diagram follows:



There are 10 different trips. (Note that in only 4 of them are all nine points covered.)